Bennett & Stinespring, Together at Last

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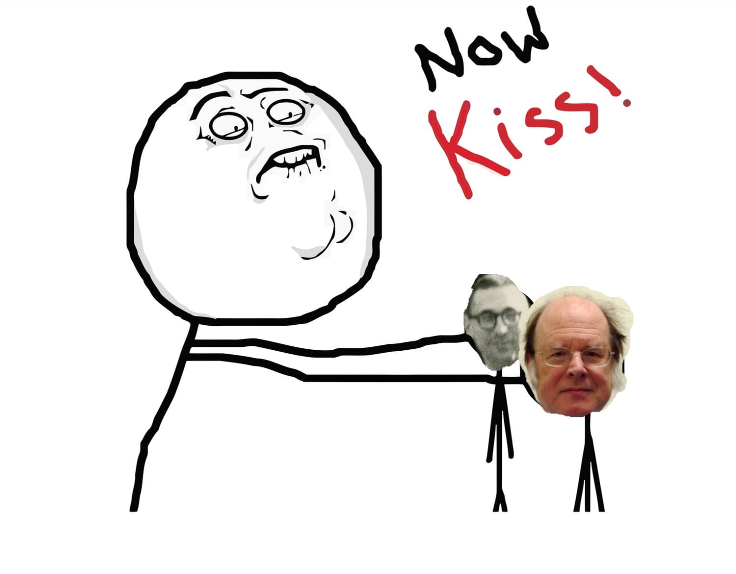
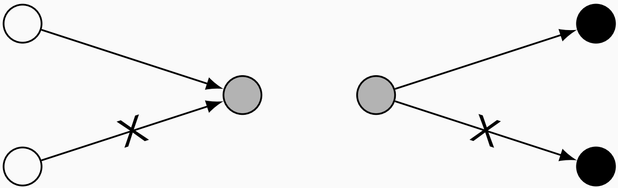

Now
Kiss!

Reversible computation



In *(forward) deterministic computation*, the current computation state *uniquely* determines the *next* computation state.

In *backward deterministic computation*, the current computation state *uniquely* determines the *previous* computation state.

Reversible computation is *forward and backward deterministic*.

Examples: Reversible Turing-machines, quantum circuits without measurement.

Reversible dynamics	Irreversible dynamics
PInj of sets and partial injective functions	**Pfn** of sets and partial functions
Unitary of f.d. Hilbert spaces and unitaries	**CPTP** of f.d. Hilbert spaces and quantum channels

What is the relationship between these?

Theorem: Any deterministic 1-tape Turing machine can be simulated by a reversible 3-tape Turing machine.

This theorem, known as *Bennett's method*, requires us to disregard any extraneous data on the two extra tapes.

Stage	Working tape	History tape	Output tape
Compute	Input	–	–
	Output	History	–
Copy	Output	History	Output
Uncompute	Input	–	Output

Stinespring's theorem

Theorem: Every quantum channel $H_A \xrightarrow{\Lambda} H_B$ is of the form

$$\Lambda(\rho_A) = \mathrm{tr}_E(U \rho_A U^\dagger)$$

for an isometry U.

In other words, every channel can be thought of as a two-step process of a reversible channel and a channel hiding the extraneous output (the environment):

$$H_A \xrightarrow{U(-)U^\dagger} H_B \otimes H_E \xrightarrow{\mathrm{tr}_E(-)} H_B$$

Uniting Bennett and Stinespring

Observation: Both Bennett's method and Stinespring's theorem rely on the ability to *hide* extraneous outputs.

Working hypothesis: Irreversible computation (whether classical or quantum) is reversible computation with hiding.

In monoidal categories, hiding is realised by projections

$$A \xleftarrow{\pi_1} A \otimes B \xrightarrow{\pi_2} B$$

and a sufficient condition for the presence of these is that the unit I is terminal (it is an *affine monoidal category*).

Problem: $\mathbf{Pfn}$ has hiding through projections, but the unit is *not* terminal, though it is "essentially terminal" – there is not a *unique* map $A \to I$, but there is a unique *total* one.

Monoidal restriction categories

A restriction category is a category with a *restriction structure*, a combinator

$$\frac{A \xrightarrow{f} B}{A \xrightarrow{\overline{f}} A}$$

satisfying $f \circ \overline{f} = f$ and other laws. The *restriction idempotent* $\overline{f}$ measures "how partial" f is (total maps, such as all isomorphisms, satisfy $\overline{f} = \mathrm{id}$).

Any category can be trivially made into a restriction category with $\overline{f} = \mathrm{id}$ for all f.

A restriction category has a *restriction terminal* object 1 if there is a unique *total* map $A \to 1$ for each object A.

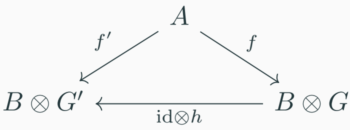
A monoidal restriction category is a restriction category which is also monoidal and satisfies $\overline{f \otimes g} = \overline{f} \otimes \overline{g}$.

The restriction affine completion

To test our hypothesis, we need to come up with a way to formally add hiding to an arbitrary monoidal restriction category $\mathbf{C}$.

We define $\mathrm{Aux}(\mathbf{C})$ as follows:

- **Objects:** Objects of $\mathbf{C}$.
- **Morphisms:** A morphism $A \to B$ is a pair of an object G and a morphism $A \to B \otimes G$ of $\mathbf{C}$, quotiented by the equivalence relation generated by the preorder defined as follows: $(f, G) \triangleleft (f', G')$ iff $\overline{f} = \overline{f'}$ and there exists $G \xrightarrow{h} G'$ in $\mathbf{C}$ such that

  commutes in $\mathbf{C}$.

The restriction affine completion

Theorem: When $\mathbf{C}$ is a monoidal restriction category so is $\mathrm{Aux}(\mathbf{C})$, and there is a monoidal restriction functor $\mathbf{C} \to \mathrm{Aux}(\mathbf{C})$.

Theorem: The monoidal unit I is restriction terminal in $\mathrm{Aux}(\mathbf{C})$.

We can show that $\mathrm{Aux}(\mathbf{C})$ is the *restriction affine completion* of $\mathbf{C}$:

Theorem: For any restriction affine monoidal category $\mathbf{D}$ and restriction monoidal functor $\mathbf{C} \xrightarrow{F} \mathbf{D}$, there is a *unique* restriction affine monoidal functor $\mathrm{Aux}(\mathbf{C}) \xrightarrow{\hat{F}} \mathbf{D}$ making the diagram below commute.

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathrm{Aux}(\mathbf{C}) \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle \hat{F}} \\ & & \mathbf{D} \end{array}$$

Theorem (Huot & Staton): $\mathrm{Aux}(\mathbf{Isometry})$ is restriction monoidally equivalent to $\mathbf{CPTP}$.

However, interestingly, $\mathrm{Aux}(\mathbf{PInj})$ is *not* equivalent to $\mathbf{Pfn}$!

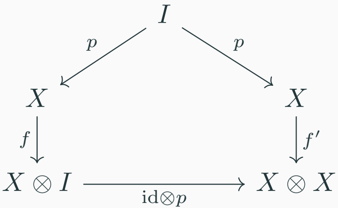
We would want to identify morphisms $A \xrightarrow{(f,G)} B$ and $A \xrightarrow{(f',G')} B$ in $\mathrm{Aux}(\mathbf{PInj})$ if in $\mathbf{Pfn}$, $\pi_1 \circ f = \pi_1 \circ f'$, but this is not the case.

Consider $X \xrightarrow{f} X \otimes I$ given by $f(x) = (x, *)$, and $X \xrightarrow{f'} X \otimes X$ given by $f'(x) = (x, x)$. Clearly $\pi_1 \circ f = \pi_1 \circ f'$, but $X \xrightarrow{(f,I)} X$ and $X \xrightarrow{(f',X)} X$ are *not* equivalent in $\mathrm{Aux}(\mathbf{PInj})$ unless $X \cong I$.

In other words, unlike **Isometry**, **PInj** has "too much" freedom in choice of reversibilisation.

M. Huot, S. Staton, Universal Properties in Quantum Theory. In *Proceedings of the 15th International Conference on Quantum Physics and Logic* (QPL 2018), EPTCS 287, 2019.

However, notice that at each point $I \xrightarrow{p} X$, it *is* the case that $I \xrightarrow{p} X \xrightarrow{(f,I)} X$ and $I \xrightarrow{p} X \xrightarrow{(f',X)} X$ are equivalent in $\mathrm{Aux}(\mathbf{PInj})$. We can always choose p itself to mediate, as in



This turns out to be a general problem in $\mathrm{Aux}(\mathbf{PInj})$: It is not well-pointed, yet $\mathbf{Pfn}$ is. So let's make it well-pointed.

Given a restriction category $\mathbf{C}$ with a restriction terminal object I, we form a new category $\mathrm{Ext}(\mathbf{C})$ as follows:

- **Objects:** Objects of $\mathbf{C}$.
- **Morphisms:** Morphisms of $\mathbf{C}$ quotiented by the equivalence $f \sim f'$ iff $f \circ p = f' \circ p$ for all $I \xrightarrow{p} X$, where $X \xrightarrow{f} Y$ and $X \xrightarrow{f'} Y$.

Theorem: When $\mathbf{C}$ is a restriction category with a restriction terminal object so is $\mathrm{Ext}(\mathbf{C})$, and there is a functor $\mathbf{C} \to \mathrm{Ext}(\mathbf{C})$.

Indeed, it can be shown that this also has a universal property (details in paper).

With this additional step, Bennett and Stinespring are together at last:

Theorem: $\mathrm{Ext}(\mathrm{Aux}(\mathbf{Isometry})) \cong \mathbf{CPTP}$.

Theorem: $\mathrm{Ext}(\mathrm{Aux}(\mathbf{PInj})) \cong \mathbf{Pfn}$.

But wait, we wanted to know the relationship between $\mathbf{Unitary}$ and $\mathbf{CPTP}$, not $\mathbf{Isometry}$ and $\mathbf{CPTP}$!

For this, we'll need …

The (restriction) coaffine completion is given by $\mathrm{Inp}(\mathbf{C}) = \mathrm{Aux}(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$.

Both $\mathbf{PInj}$ and $\mathbf{Unitary}$ are rig categories, and we can use the dual completion to make the unit of the direct sum $\oplus$ initial.

This completes $\mathbf{Unitary}$ to $\mathbf{Isometry}$, but is invariant on $\mathbf{PInj}$ (as the unit of the sum is already initial):

Theorem: $\mathrm{Inp}_{\oplus}(\mathbf{Unitary}) \cong \mathbf{Isometry}$ but $\mathrm{Inp}_{\oplus}(\mathbf{PInj}) \cong \mathbf{PInj}$.

Putting all of these together, we get

Theorem: $\mathrm{Ext}(\mathrm{Aux}_{\otimes}(\mathrm{Inp}_{\oplus}(\mathbf{Unitary}))) \cong \mathbf{CPTP}$.

Theorem: $\mathrm{Ext}(\mathrm{Aux}_{\otimes}(\mathrm{Inp}_{\oplus}(\mathbf{PInj}))) \cong \mathbf{Pfn}$.

M. Huot, S. Staton. Quantum channels as a categorical completion. In *34th Annual ACM/IEEE Symposium on Logic in Computer Science* (LICS 2019), IEEE, 2019

We can also essentially recover $\mathbf{PInj}$ and $\mathbf{Unitary}$ from $\mathbf{Pfn}$ and $\mathbf{CPTP}$ respectively, as their *cofree inverse categories* $\mathrm{Inv}(-)$ (details in paper).

Theorem: $\mathrm{Inv}(\mathbf{Pfn}) \cong \mathbf{PInj}$.

Theorem: $\mathrm{Inv}(\mathbf{CPTP}) \cong \mathbf{Unitary}_p$.

In the above, $\mathbf{Unitary}_p$ is the category of finite dimensional Hilbert spaces and unitaries identified up to global phase.

In summary

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Wait, it's all reversible?
Always has been

Thank you!